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Extending the weak-equilibrium condition for algebraic Reynolds stress models to rotating and curved flows

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Weis & Hutter (*J. Fluid Mech.* vol. 476, 2003, p. 63) obtained an implicit algebraic Reynolds stress model from a differential Reynolds stress transport equation valid in an arbitrarily rotating time-dependent coordinate frame (relative to an inertial system). Although the form of this implicit algebraic equation differed from previous implicit forms, its correctness was argued based on the objective tensor form of the implicit algebraic equation. It is shown here that such conclusions based on simple coordinate frame transformations are incomplete, and that additional considerations taking into account flow rotation and curvature, for example, are necessary. By properly accounting for both the arbitrary motions of the observer coordinate frame as well as the motion of the flow itself, it is shown that previous formulations and application of the weak-equilibrium condition are correct in contrast to the results of Weis & Hutter.

1. Introduction

Over the last decade, there has been widespread use of explicit algebraic stress models in computing turbulent flow fields. The origin of the implicit models from which the explicit ones have been derived has been attributed to Rodi (1972, 1976). The basis for the explicit forms commonly in use today is due to Pope (1975). Further details about these explicit algebraic models can be found elsewhere (Girimaji 1996; Gatski & Jongen 2000; Wallin & Johansson 2000; Gatski & Rumsey 2002). In inertial frames of reference, there is no ambiguity in the assumptions required to obtain the implicit algebraic Reynolds stress equations from the corresponding differential form. These assumptions are that the material derivative of the Reynolds stress anisotropy tensor b_{ij} is zero and that the anisotropy of the turbulent transport and viscous diffusion term tensor $\mathscr{D}_{ij}$ is proportional to the Reynolds stress itself. These two assumptions (conditions) can be written as,

$$\frac{\mathrm{D}b_{ij}}{\mathrm{D}t} = 0, \tag{1.1}$$

where $\mathrm{D}/\mathrm{D}t = \partial/\partial t + U_k \partial/\partial x_k$, and

$$\mathscr{D}_{ij} = \frac{\tau_{ij}}{2K}\mathscr{D}_{kk}, \tag{1.2}$$

respectively. Here, the Reynolds stress anisotropy is defined as

$$b_{ij} = \frac{\tau_{ij}}{2K} - \frac{\delta_{ij}}{3}, \tag{1.3}$$

where $\tau_{ij} = \overline{u_i u_j}$ are the Reynolds stresses and $K = \tau_{kk}/2$, and the turbulent transport and viscous diffusion term tensor is given by

$$\mathscr{D}_{ij} = -\frac{\partial}{\partial x_k}(\overline{u_i u_j u_k} + \overline{p u_i}\delta_{jk} + \overline{p u_j}\delta_{ik}) + \nu\nabla^2\tau_{ij}. \tag{1.4}$$

(The convention that capital letters are associated with mean values and lower-case letters are associated with the turbulent fluctuations is adopted here.) The focus of this study is on the suitability of the equilibrium condition associated with (1.1).

In flows described relative to an inertial frame of reference and in flows where the local rotation rate of the fluid is fixed along a streamline (that is, no curvature and/or flow rotation effects present), the conditions given in (1.1) and (1.2) can be applied unambiguously. However, in formulating algebraic Reynolds stress models in flows relative to non-inertial frames or in flows where local rotation rate effects of the fluid are not fixed (that is, curvature and/or flow rotation effects present), alternative conditions must be considered. Specifically, while (1.1) applies in the former case, alternative forms must be considered for the latter more complex flow case. This was realized over twenty years ago by Rodi & Scheuerer (1983) and is the subject of the analysis reported here.

2. Transformation properties of the Reynolds stress anisotropy transport equations

It is necessary at the outset of discussions on transformation properties to establish the fundamental relations and then to carry those through in transforming any equations. Otherwise, notational ambiguities arise and it is very difficult to cross-reference with other related work.

Let the base Eulerian system (the frame in which the observer is fixed) be given by the rectangular coordinates x_i. Now consider the rectangular coordinates x_i^* of a point in a frame of reference in arbitrary time-dependent motion relative to the fixed (inertial) Eulerian frame identified with x_i. The transformation rule between these two frames is simply given by

$$x_i^*(t^*) = Q_{ij}[x_j + x_{0j}], \tag{2.1}$$

where $\boldsymbol{Q} = \boldsymbol{Q}(t)$ is a time-dependent proper orthogonal tensor ($\boldsymbol{Q}\boldsymbol{Q}^T = \boldsymbol{I}$, $\det \boldsymbol{Q} = +1$, $\boldsymbol{I}$ is the identity tensor), $x_{0j} = x_{0j}(t)$ is a time-dependent displacement vector, and $t^* = t + t_0$ (t_0 is an arbitrary constant time shift). Without loss of generality in the present analysis, translational accelerations are not considered ($\dot{x}_{0j} = \text{const}$).

It is well established that the Reynolds stress anisotropy tensor b_{ij} is a function of both the strain rate ($\boldsymbol{S}$) and rotation rate ($\boldsymbol{W}$) tensors such that

$$b_{ij} = b_{ij}(\boldsymbol{S}, \boldsymbol{W}, \tau), \tag{2.2}$$

where

$$S_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right), \qquad W_{ij} = \frac{1}{2}\left(\frac{\partial U_i}{\partial x_j} - \frac{\partial U_j}{\partial x_i}\right), \tag{2.3}$$

and τ is some turbulent time scale. (Note that throughout there will be an interchange between tensor and matrix notation to simplify the notational burden where possible.)

In addition, it is known that the anisotropy tensor b_{ij} is objective (Speziale 1979), that is, it is the same no matter which coordinate frame an observer measures it in. Such objective tensor quantities transform from one frame to the other in the manner of Eringen (1980)

$$b^*_{ij} = Q_{ik}Q_{jl}b_{kl}. \tag{2.4}$$

Note that from (2.2) it follows that b_{ij} should be a function of objective quantities as well. Relative to Euclidean transformations, the strain rate tensor S_{ij} is objective while the rotation rate tensor W_{ij} is not. Only under a Galilean transformation (a translatory transformation) will W_{ij} itself be objective. With the definition of W_{ij} in (2.3) in terms of the proper definition of the velocity ($U_i \equiv \partial X_i/\partial t$), the S_{ij} and W_{ij} tensors transform as

$$S_{ij} = Q_{ki}S^*_{kl}Q_{lj}, \tag{2.5a}$$

$$W_{ij} = Q_{ki}W^*_{kl}Q_{lj} + \dot{Q}_{ki}Q_{kj}, \tag{2.5b}$$

where an overdot denotes the time derivative. Equation (2.5*b*) can be rewritten in the form

$$W_{ij} = Q_{ki}(W^*_{kl} + \Omega^*_{kl})Q_{lj}. \tag{2.6}$$

$\Omega_{ij} = \dot{Q}_{ki}Q_{kj} = \epsilon_{jik}\omega_k$ is the rotation rate tensor associated with the angular rotation rate vector ω_k of the x^*_i coordinates and $\Omega^*_{ij} = Q_{ik}\Omega_{kl}Q_{jl}$ is the rotation rate expressed in the x^*_i system.

Equation (2.6) also shows that the rotation rate tensor W^*_{ij} can be made objective by adding a measure of the non-inertial frame rotation rate Ω^*_{ij}. This modification of W^*_{ij} is not surprising since objective variables are quantities independent of the motion of the observer (Eringen 1980). Astarita (1979) provides a simple proof of the objective character of $W^*_{ij} + \Omega^*_{ij}$. Although the functional dependency given in (2.2) holds for inertial frames as well as other frames under a Galilean transformation, it will be shown shortly that under the more general Euclidean transformation group, a corresponding relation, in terms of objective variables, can be derived.

Since the goal of this analysis is to ascertain the proper set of conditions on which to base the form of an algebraic Reynolds stress model, it is necessary to first examine the Reynolds stress anisotropy equation which can be written as

$$\frac{\mathrm{D}b_{ij}}{\mathrm{D}t} = -\left(\frac{\mathscr{P}}{\varepsilon} - 1\right)\frac{b_{ij}}{\tau} - \tfrac{2}{3}S_{ij} - \left(b_{ik}S_{kj} + S_{ik}b_{kj} - \tfrac{2}{3}[\boldsymbol{b}:\boldsymbol{S}]\delta_{ij}\right) + (b_{ik}W_{kj} - W_{ik}b_{kj}) + \frac{\Pi_{ij}}{2K}, \tag{2.7}$$

where Π_{ij} is the pressure–strain rate correlation, ε is the isotropic scalar dissipation rate, and $\mathscr{P} = -2Kb_{mn}S_{nm}$ is the production of turbulent kinetic energy. Here, the effect of turbulent transport and diffusion of the anisotropy has been neglected.

Numerous models have been proposed for the pressure–strain rate correlation with the functional dependency generally given by $\Pi_{ij} = \Pi_{ij}(b_{kl}, S_{kl}, W_{kl}, \tau)$. For the present purposes, only models linear in b_{kl}, S_{kl}, and W_{kl} need be considered and these take the form,

$$\frac{\Pi_{ij}}{K} = \alpha_1\frac{b_{ij}}{\tau} + \alpha_2 S_{ij} + \alpha_3\left(b_{ik}S_{kj} + S_{ik}b_{kj} - \tfrac{2}{3}[\boldsymbol{b}:\boldsymbol{S}]\delta_{ij}\right) + \alpha_4(b_{ik}W_{kj} - W_{ik}b_{kj}). \tag{2.8}$$

The question now is what form does the Reynolds stress anisotropy equation (2.7), and the modelled pressure–strain rate correlation take in the non-inertial frame. Let us write the Reynolds stress anisotropy equation in the following form

$$\frac{\mathrm{D}b_{ij}}{\mathrm{D}t} = f_{ij}(b_{kl}, S_{kl}, W_{kl}), \tag{2.9}$$

for a clear illustration of the different transformation properties. A formal transformation to the x_i^* system results in

$$\frac{\mathrm{D}b^*_{ij}}{\mathrm{D}t} - (b^*_{ik}\Omega^*_{kj} - \Omega^*_{ik}b^*_{kj}) = f^*_{ij}(b^*_{kl}, S^*_{kl}, W^*_{kl} + \Omega^*_{kl}). \tag{2.10}$$

Note that the complete left-hand side is the advection of the anisotropy expressed in the x_i^* system and it is not correct to label only the first differential part, $\mathrm{D}b^*_{ij}/\mathrm{D}t \equiv \partial b^*_{ij}/\partial t + U^*_k \partial b^*_{ij}/\partial x^*_k$, as the advection or time derivative in the x_i^* system. $\mathrm{D}b^*_{ij}/\mathrm{D}t$ only represents the advection or time derivative of the individual components of the transformed anisotropy. Equations (2.10), (2.7) and (2.8) can be combined to

$$\begin{aligned}\frac{\mathrm{D}b^*_{ij}}{\mathrm{D}t} - (b^*_{ik}\Omega^*_{kj} - \Omega^*_{ik}b^*_{kj}) = &-\left(\frac{\mathscr{P}}{\varepsilon} - 1 - \tfrac{1}{2}\alpha_1\right)\frac{b^*_{ij}}{\tau} - \left(\tfrac{2}{3} - \tfrac{1}{2}\alpha_2\right)S^*_{ij} \\ &- \left(1 - \tfrac{1}{2}\alpha_3\right)\left(b^*_{ik}S^*_{kj} + S^*_{ik}b^*_{kj} - \tfrac{2}{3}[\boldsymbol{b}^* : \boldsymbol{S}^*]\delta_{ij}\right) \\ &+ \left(1 + \tfrac{1}{2}\alpha_4\right)(b^*_{ik}\overline{W}^*_{kj} - \overline{W}^*_{ik}b^*_{kj}),\end{aligned} \tag{2.11}$$

where the objective absolute vorticity tensor is

$$\overline{W}^*_{ij} = W^*_{ij} + \Omega^*_{ij}. \tag{2.12}$$

Equation (2.11) is consistent with the assertion made following (2.4) that the anisotropy tensor should only depend on objective quantities, and is also consistent with forms that have been arrived at previously (e.g. Speziale 1979; Wallin & Johansson 2000; Wies & Hutter 2003). These results show that the Reynolds stress anisotropy equation given by (2.7) is not form-invariant with a change of coordinate frame under the Euclidean group. Only under the Galilean group ($\boldsymbol{\Omega}^* = 0$) would (2.7) be form-invariant (Speziale 1979). However, (2.7) when transformed under the Euclidean group can be rewritten in terms of objective variables such that the functional relation given by (2.2) can be extended to the more general form

$$b^*_{ij} = b^*_{ij}(\boldsymbol{S}^*, \overline{\boldsymbol{W}}^*, \tau). \tag{2.13}$$

Although these tensor manipulations provide a formalism with which to discuss the various forms of the transport equations in different coordinate frames as well as the proper functional dependencies, they do not provide any information on the behaviour of the solutions of the equations themselves. As pointed out in § 1, the implicit form of an algebraic Reynolds stress model depends on a weak-equilibrium condition associated with the Reynolds stress anisotropy. In flows with minor effects of rotation or mean flow curvature the relevant condition was $\mathrm{D}b_{ij}/\mathrm{D}t = 0$. This is the classical weak-equilibrium condition as proposed by Pope. This obviously results in neglecting the complete left-hand side of (2.10) and is also the most obvious way of obtaining an objective relation for the anisotropy tensor.

In the next section, this condition is generalized to flow fields where more complex non-inertial effects are present.

3. Equilibrium conditions in rotating and curved flows

The tensorial transformation properties of the Reynolds stress anisotropy equation just established were kinematic in nature and did not reflect in any way on the dynamic state of the turbulent flow field. In this section, a general set of equilibrium conditions on the turbulence dynamics will be extracted using the transformation properties established in the previous section.

Consider the time-dependent motion of a turbulent flow relative to some Eulerian frame x_i. Two such examples are a turbulent flow rotating uniformly relative to an inertial x_i frame, and a turbulent flow with arbitrary mean streamline curvature relative to an inertial x_i frame. The question that now arises in formulating an implicit algebraic Reynolds stress equation is what form does the weak-equilibrium condition take in such flows. It is not difficult to envisage that the standard form ($\mathrm{D}b_{ij}/\mathrm{D}t = 0$) used in the simple rectilinear flows relative to an inertial frame will not be satisfied.

Let the coordinate system that follows the flow be given by the rectangular coordinates $x_i^\dagger$, and let the arbitrary time-dependent rotation of the $x_i^\dagger$ coordinates be represented by $\omega^{(r)}$.

It is important to recognize that there is a distinction between a coordinate frame in which an observer is fixed (x_i^* system), and a coordinate frame embedded within the fluid ($x_i^\dagger$ system). Each can be undergoing arbitrary motion relative to one another as well as to some (common) Eulerian, inertial frame.

Let the transformation to the $x_i^\dagger$ system be given by

$$x_i^\dagger = T_{ij}[x_j + x_{0j}], \tag{3.1}$$

where $\boldsymbol{T} = \boldsymbol{T}(t)$ is a time-dependent proper orthogonal tensor ($\boldsymbol{TT}^T = \boldsymbol{I}$, det $\boldsymbol{T} = +1$), and x_{0j} is a displacement vector introduced in (2.1). The rotation rate tensor $\Omega_{ij}^{(r)} = \dot{T}_{ki}T_{kj} = \epsilon_{jik}\omega_k^{(r)}$ is associated with the angular rotation rate vector $\omega_k^{(r)}$ of the $x_i^\dagger$ coordinates, or the rotation rate vector of the flow.

Consider now the functional relationship for the Reynolds stress anisotropy tensor given in (2.9) which is deduced from its modelled transport equation. Since the $f_{ij}\,(b_{kl}, S_{kl}, W_{kl})$ is an isotropic function of it arguments, the Reynolds stress anisotropy equation in (2.9) can be written in the $x_i^\dagger$ system as

$$\frac{\mathrm{D}b_{ij}^\dagger}{\mathrm{D}t} - \left(b_{ik}^\dagger \Omega_{kj}^{(r)\dagger} - \Omega_{ik}^{(r)\dagger} b_{kj}^\dagger\right) = f_{ij}^\dagger\left(b_{kl}^\dagger, S_{kl}^\dagger, W_{kl}^\dagger + \Omega_{kl}^{(r)\dagger}\right). \tag{3.2}$$

The first term $\mathrm{D}b_{ij}^\dagger/\mathrm{D}t \equiv \partial b_{ij}^\dagger/\partial t + U_k^\dagger \partial b_{ij}^\dagger/\partial x_k^\dagger$ represents the advection of the individual components of the anisotropy tensor in the $x_i^\dagger$ coordinate system (that is the system embedded in the flow) and, if the flow is homogeneous following the mean motion, then $\mathrm{D}b_{ij}^\dagger/\mathrm{D}t = 0$ identically.

As noted previously, the $x_i^\dagger$ system can be independent of the coordinate system of the observer (the x_i^* system). Thus, it is straightforward to transform (3.2) back to the inertial system x_i

$$T_{ki}\frac{\mathrm{D}b_{kl}^\dagger}{\mathrm{D}t}T_{lj} - \left(b_{ik}\Omega_{kj}^{(r)} - \Omega_{ik}^{(r)}b_{kj}\right) = f_{ij}(b_{kl}, S_{kl}, W_{kl}). \tag{3.3}$$

Irrespective of the coordinate system that one is in, the correct form of the weak-equilibrium assumption must be that $\mathrm{D}b_{ij}^\dagger/\mathrm{D}t = 0$. This means that the first term on the left-hand side of (3.3) must vanish, and that the resulting implicit algebraic equation for b_{ij} in the inertial frame will be given by

$$f_{ij}\left(b_{kl}, S_{kl}, W_{kl}\right) + \left(b_{ik}\Omega_{kj}^{(r)} - \Omega_{ik}^{(r)}b_{kj}\right) = 0. \tag{3.4}$$

This result can be generalized further. Once again, consider the observer in the x_i^* frame. It is straightforward to transform (3.3) to the x_i^* system by applying the transformation operator $\boldsymbol{Q}$ to this equation. This results in an equation for the anistropy tensor b_{ij}^* given by

$$\boldsymbol{QT}^T \frac{\mathrm{D}\boldsymbol{b}^\dagger}{\mathrm{D}t}\boldsymbol{TQ}^T - \left(\boldsymbol{b}^*\boldsymbol{\Omega}^{(r)^*} - \boldsymbol{\Omega}^{(r)^*}\boldsymbol{b}^*\right) = \boldsymbol{f}^*(\boldsymbol{b}^*, \boldsymbol{S}^*, \boldsymbol{W}^* + \boldsymbol{\Omega}^*), \tag{3.5a}$$

or

$$\boldsymbol{QT}^T \frac{\mathrm{D}\,\boldsymbol{TQ}^T\boldsymbol{b}^*\boldsymbol{QT}^T}{\mathrm{D}t}\boldsymbol{TQ}^T - \left(\boldsymbol{b}^*\boldsymbol{\Omega}^{(r)^*} - \boldsymbol{\Omega}^{(r)^*}\boldsymbol{b}^*\right) = \boldsymbol{f}^*(\boldsymbol{b}^*, \boldsymbol{S}^*, \boldsymbol{W}^* + \boldsymbol{\Omega}^*). \tag{3.5b}$$

(To simplify the form of the equation, matrix notation is used here.) Observe that the rotation rate tensor $\boldsymbol{\Omega}^{(r)^*}$ on the left-hand side, in general, differs from the rotation rate tensor $\boldsymbol{\Omega}^*$ added to the vorticity tensor on the right-hand side of (3.5*a*) or (3.5*b*). The former represents a measure of the rotation rate of the flow, while the latter represents the rotation rate of the observer.

Once again, applying the weak-equilibrium assumption $\mathrm{D}b_{ij}^\dagger/\mathrm{D}t = 0$ yields an implicit algebraic equation for $\boldsymbol{b}^*$ in the x_i^* system

$$\boldsymbol{f}^*(\boldsymbol{b}^*, \boldsymbol{S}^*, \boldsymbol{W}^* + \boldsymbol{\Omega}^*) + \left(\boldsymbol{b}^*\boldsymbol{\Omega}^{(r)^*} - \boldsymbol{\Omega}^{(r)^*}\boldsymbol{b}^*\right) = 0. \tag{3.6}$$

Note that if the x_i^* system of the observer coincides with the $x_i^\dagger$ system embedded in the fluid flow ($\Omega_{ij}^{(r)} = \Omega_{ij}^*$), then (3.6) reduces to

$$\boldsymbol{f}^*(\boldsymbol{b}^*, \boldsymbol{S}^*, \boldsymbol{W}^* + \boldsymbol{\Omega}^*) + (\boldsymbol{b}^*\boldsymbol{\Omega}^* - \boldsymbol{\Omega}^*\boldsymbol{b}^*) = 0. \tag{3.7}$$

This is the situation that commonly arises in the study of turbulent rotating flows such as rotating homogeneous shear flow and rotating channel flow. In these flows, the turbulence is analysed relative to a coordinate frame used by an observer that is moving at the same rotation rate as the mean flow (e.g. Speziale & Mac Giolla Mhuiris 1989; Speziale, Gatski & Mac Giolla Mhuiris 1990; Jongen, Machiels & Gatski 1998). A long time equilibrium state exists that is characterized by fixed points given by $b_{ij}^* = \text{const}$. This situation corresponds to the case discussed previously where the coordinate system used by the observer coincides with the $x_i^\dagger$ system embedded in the fluid. In this case, the equation governing the turbulence is given by (3.7). This equation shows a direct dependence on the system rotation rate that would then yield different equilibrium values ($t \to \infty$) for b_{ij}^* for different rotation rates. Obviously, relation equation (3.7) is not Euclidean invariant, since it is only valid for the specific choice that the (rotational) motion of the observer coincides with the (rotational) motion of the flow.

The correctness of the extended weak-equilibrium condition $\mathrm{D}b_{ij}^\dagger/\mathrm{D}t = 0$ derived here in (3.3) and resulting in (3.7), is clearly illustrated by computing a fully developed rotating channel flow. Using the full differential Reynolds stress model (DRSM) proposed by Wallin & Johansson (2002), it is shown in figure 1 that the EARSM derived using the extended weak equilibrium condition predicts a velocity profile that closely approximates the corresponding DRSM while the prediction from the original weak equilibrium condition, and advocated by Weis & Hutter (2003), is clearly not applicable to such flows. Similar results were also obtained by Jongen

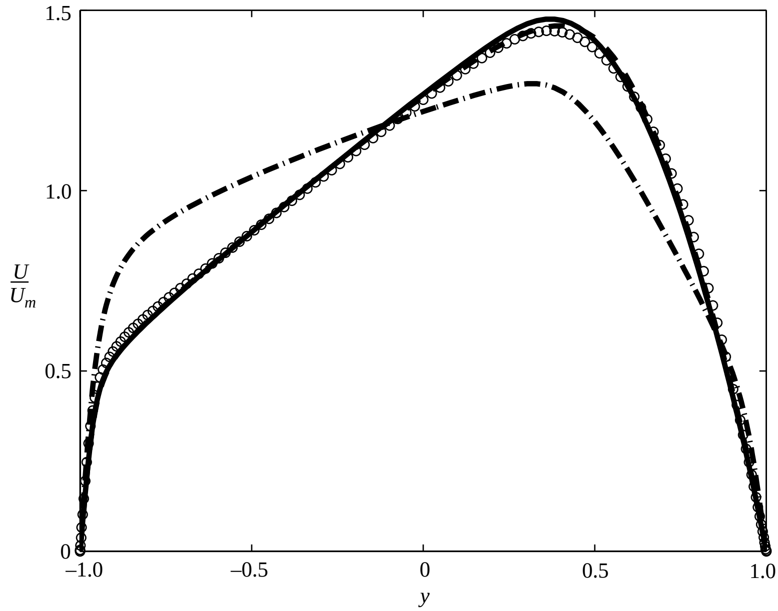

FIGURE 1. Mean velocity distribution across rotating channel flow for $Ro = 0.77$ and $Re_\tau = 180$. Computations using EARSM applying $\mathrm{D}b^{\dagger}_{ij}/\mathrm{D}t = 0$ (——) in (3.3) compared to the corresponding DRSM (– – –) and the EARSM applying $\mathrm{D}b_{ij}/\mathrm{D}t = 0$ (– · –). ○, DNS of Alvelius (1999).

et al. (1998) using the simulation data of Lamballais, Lesieur & Métais (1996) and a different EARSM. It should be mentioned that a standard eddy-viscosity model, without explicit rotation corrections, will predict a symmetric velocity profile.

A more general situation arises in the case of curved turbulent flows. In these flows, the turbulence is commonly analysed relative to an observer fixed in an inertial frame x_i. A measure of flow curvature is deduced from the coordinate frame $x_i^{\dagger}$ embedded in the flow. There have been several Galilean invariant proposals put forward for identifying $x_i^{\dagger}$ (e.g. Girimaji 1997; Sjögren 1997; Gatski & Jongen 2000; Wallin & Johansson 2002) and these have been analysed by Wallin & Johansson (2002) and Hellsten (2002). For example, the proposal by Gatski & Jongen (2000), later extended to three-dimensional flows by Wallin & Johansson (2002), uses a $x_i^{\dagger}$ coordinate frame in principle aligned with the principal axes of the strain rate tensor S_{ij}. In these flows, a fixed point is reached that is characterized by the condition $\boldsymbol{b}^{\dagger} = \text{const.}$, and (3.4) applies. An example of a curved flow solution where the effect of imposing the correct weak-equilibrium solution is analysed is given by Rumsey, Gatski & Morrison (2000) for the case of turbulent flow in a U-bend.

4. Concluding remarks

The results in the previous section directly contradict those of Weis & Hutter (2003) who base their conclusions on a rationale that fails to account for the rotation or curvature of the flow itself. As such, their analysis stops with the long-established form of the Reynolds stress anisotropy equation given in (2.11). At this stage, they argue that it is incorrect for the observer in the x_i^* frame to impose a condition on the Reynolds stress anisotropy tensor (i.e. $\mathrm{D}b^*_{ij}/\mathrm{D}t = 0$) that differs from an observer in the x_i inertial frame (i.e. $\mathrm{D}b_{ij}/\mathrm{D}t=0$). This is the rational conclusion if, and only if, the anisotropy is forced to be independent of $\boldsymbol{\Omega}^{(r)}$ as proposed by Rodi (1972, 1976). Thus, the turbulence is modelled to be dependent on $\boldsymbol{S}$ and $\boldsymbol{W}$, the strain rate and (absolute) vorticity tensors, only. The argument then follows that the proper weak-equilibrium condition to impose is $\mathrm{D}b^*_{ij}/\mathrm{D}t-(b^*_{ik}\Omega^*_{kj}-\Omega^*_{ik}b^*_{kj})=0$. Weis & Hutter (2003) further argue that with this condition, the resulting implicit algebraic equation for b^*_{ij} is invariant since it is composed solely of objective tensors.

However, once again, the turbulence is indeed also dependent on the rotation or curvature of the flow itself represented by $\boldsymbol{\Omega}^{(r)}$. Weis & Hutter (2003) claim that this is inconsistent with Euclidean invariance. In the previous section, it has been clearly shown that $\boldsymbol{\Omega}^{(r)}$ indeed can be accounted for, where the correct invariance properties are preserved. Thus, the modelled Reynolds stress tensor is independent of the rotation rate of the observer.

In the notation used here, the conclusions of Weis & Hutter (2003) lead to an incomplete form for the implicit algebraic Reynolds stress anisotropy equation

$$\boldsymbol{f}^*(\boldsymbol{b}^*, \boldsymbol{S}^*, \boldsymbol{W}^*+\boldsymbol{\Omega}^*)=0. \tag{4.1}$$

When compared with the forms for the implicit algebraic equations derived here, (3.6) and (3.7), when the motion of the flow itself is taken into account, it is evident that the results obtained from the two formulations would differ significantly. Additionally, the notion of invariant modelling employed by Weis & Hutter (2003) is not consistent with the behaviour of the differential Reynolds stress transport equations. It is well known that these equations are not form-invariant under arbitrarily rotating time-dependent (Euclidean) transformations. Thus, there is no reason why this condition of frame invariance should be imposed on a set of implicit algebraic equations that are intended to replicate, as closely as possible, the predictive capabilities of the differential forms. The condition that should be imposed is that the implicit algebraic equations retain as closely as possible the frame-invariance properties of the differential form.

It has been shown here how the weak-equilibrium condition required in the development of implicit algebraic stress models is consistently imposed on equations describing turbulent flows with rotation or curvature effects. Although these forms have been derived and used previously, Weis & Hutter (2003) have recently developed an alternative weak-equilibrium condition that directly contradicts the established forms and is based on an incomplete analysis of the transformation properties of the governing equations. The analysis presented here is intended to clarify the assumptions and detail the steps used in arriving at this well-established weak-equilibrium condition and its application in rotating and curved flows.

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